Application of the Weierstrass Elliptic Expansion Method to the Long-Wave and Short-Wave Resonance Interaction System

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With the aid of symbolic computation, nine families of new doubly periodic solutions are obtained for the (2+1)-dimensional long-wave and short-wave resonance interaction (LSRI) system in terms of the Weierstrass elliptic function method. Moreover Jacobian elliptic function solutions and solitary wave solutions are also given as simple limits of doubly periodic solutions.

Key words: Weierstrass Elliptic Expansion Method; LSRI System.
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1. Introduction

It is well known that the elliptic functions including Jacobian elliptic functions and Weierstrass elliptic functions, are closely related to nonlinear differential equations [1]. Moreover many nonlinear evolution equations have been shown to possess elliptic function solutions [2 – 4] and Jacobian elliptic function solutions include not only solitary wave solutions but more types of solutions depending on other different modulus. In addition, the Jacobian elliptic functions $\mathrm{sn}(\xi;m), \mathrm{cn}(\xi;m), \mathrm{dn}(\xi;m)$ can be expressed by the unified Weierstrass elliptic function $\wp(\xi;g_2,g_3)$. Therefore it is of important significance to investigate Weierstrass elliptic function solutions of nonlinear wave equations. There exist some transformations to study Weierstrass elliptic function solutions of nonlinear wave equations. In [5], many important nonlinear wave equations arising from nonlinear science are chosen to illustrate the Weierstrass elliptic function expansion method such as the new integrable Davey-Stewartson-type equation, the (2+1)-dimensional modified Korteweg-de Vries equation, the generalized Hirota equation, the (2+1)-dimensional modified Novikov-Veselov equations, and the coupled Klein-Gordon equation.

In this paper, enlightened by the idea in [5, 6], we attempt to develop an algorithm in terms of the Weierstrass elliptic function $\wp(\xi;g_2,g_3)$ to seek new types of doubly periodic solutions of the (2+1)-dimensional equation describing the interaction of a long wave with a packet of short waves. Such process can arise in fluid mechanics. If L is a long interfacial wave and S is a short waves packet, their interaction on the (x,y)-plane is described by the system [7]

$$\begin{aligned} &\mathrm{i}\partial_t S + \mathrm{i}c_\mathrm{g}\partial_x S - \beta LS - \gamma\partial_{xx}^2 S - \delta|S|^2 S = 0, \\ &\partial_t L + c_\mathrm{l}\partial_y L + \alpha\partial_x |S|^2 = 0, \end{aligned} \tag{1}$$

where c_l is the long-wave phase velocity along the y-axis, c_g is the group velocity of a packet of short waves along the x-axis, and $\alpha,\beta,\gamma,\delta$ are constant parameters of the system under consideration. Using the "resonance condition" $c_\mathrm{l} = c_\mathrm{g}$ [8], some linear coordinate transformations and scale transformation of the constants, it is possible to rewrite this system in the form

$$\mathrm{i}\partial_t S - \partial_{xx}^2 S + LS = 0, \quad \partial_y L = 2\partial_x |S|^2. \tag{2}$$

This integrable dispersive long-wave and short-wave resonance interaction (LSRI) system is an interesting topic in physics and mathematics, while the application of symbolic computation to physical and mathematical sciences appears to have a bright future. The integrability of this system has been established earlier. Therefore, a Lax pair for (1) has been constructed in [9], and an exact solution of this equation has been presented in [7]. The LSRI equation considered in [10, 11] can be written in our form by simple linear coordinate transformations. In [10] the authors have performed a deep analysis of different classes of exact solutions, such as "positon", "dromion" and "soliton", and they have derived a number of new solutions including one on a continuous wave background. The recent investigation of the above system has been performed in [11]. The authors showed that (1) possesses, the Painlevé property, and they generated an extended class of periodic Jacobian elliptic function solutions and a distinct general class of exponentially localized solutions vanishing in all directions. In these solutions, the time variable t and space variable x are separated, and the waves can only advance along the y-axis. However, in this paper, we have transformed (1) into a nonlinear ordinary differential equation [in terms of the wave variable $\xi = k(x+\lambda y+\sigma t)$] leading to particular solutions (namely plane wave solutions). These solutions advance along the x-axis as well as the y-axis. Furthermore, new plane solitary wave solutions, a special class of solutions which do not vanish along the line $\xi = 0$ for $x, y \to \infty$, are derived.

2. Introduction of the Weierstrass Elliptic Expansion Method

In the following we will simply introduce the Weierstrass elliptic expansion method and its algorithm.

Step I: Consider a given nonlinear evolution equation with a physical field u and two independent variables x, t:

$$\Phi(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0. \tag{3}$$

We make the ansatz $u(x,t) = u(\xi), \xi = k(x-\lambda t)$, which reduces (3) to a nonlinear ordinary differential equation

$$\Psi(u, u', u'', u''', \ldots) = 0, \tag{4}$$

where the prime notation represents the differentiation with respect to ξ.

Step II: Seek the power series solution of (4) in terms of the Weierstrass elliptic function

$$u(\xi) = F_i[\wp(\xi; g_2, g_3)], \quad i = 1, 2, \tag{5}$$

where

$$\begin{aligned} &F_1[\wp(\xi;g_2,g_3)] \\ &= A_0 + \sum_{i=1}^{n} \Big[\frac{A_i\wp(\xi;g_2,g_3) + B_i\wp'(\xi;g_2,g_3)}{R + P\wp(\xi;g_2,g_3) + Q\wp'(\xi;g_2,g_3)}\Big]^i, \\ &F_2[\wp(\xi;g_2,g_3)] \\ &= a_0 + \sum_{i=1}^{n} \Big(a_i\big[E_1\wp(\xi;g_2,g_3) + E_2\big]^{i/2} \\ &\qquad + b_i[E_1\wp(\xi;g_2,g_3) + E_2]^{-i/2}\Big), \end{aligned} \tag{6}$$

and $\wp(\xi; g_2, g_3)$ is the Weierstrass elliptic function satisfying the nonlinear ordinary equation

$$[\wp(\xi)]'^2 = 4\wp^3(\xi) - g_2\wp(\xi) - g_3 \tag{7}$$

or, another form,

$$[\wp(\xi)]'' = 6\wp^2(\xi) - \frac{1}{2}g_2, \tag{8}$$

with g_2, g_3 being real parameters called invariants, where the prime denotes derivative with respect to ξ.

Step III: Define a polynomial degree function as

$$D(u(\wp)) = n.$$

Thus we have

$$D\left(u^p(\wp)\left(\frac{\mathrm{d}^s u(\wp)}{\mathrm{d}\xi^s}\right)^q\right) = np + q(n+s).$$

Therefore we can determine n in (6) by the leading order analysis (or balancing the highest-order linear term and nonlinear terms).

Step IV: Substitute (6) with the known parameter n into the left-hand-side of the obtained ODE, and then take the numerator of the expression to get a polynomial equation. Set the coefficients of polynomial equal to zero to get a set of algebraic equations with respect to the unknowns.

Step V: With the aid of Maple, solve the set of algebraic equations, which may not be consistent, and finally derive the doubly periodic solutions of the given nonlinear equations by using the Weierstrass elliptic function.

3. New Doubly Periodic Solutions

We consider the following specific transformations for system (1):

$$\begin{aligned} &S(x,y,t) = S(\xi)\exp(\mathrm{i}\theta), \quad L(x,y,t) = L(\xi), \\ &\xi = k(x+\lambda y+\sigma t), \quad \theta = \alpha x + \beta y + \gamma t, \end{aligned} \tag{9}$$

where $k, \lambda, \sigma, \alpha, \beta, \gamma$ are constants to be determined later.

Then, the substitution of (9) into (1) yields

$$\begin{aligned} &(-\gamma + L(\xi) + \alpha^2)S(\xi) - \frac{\mathrm{d}^2 S(\xi)}{\mathrm{d}\xi^2}k^2 = 0, \\ &k\lambda\frac{\mathrm{d}L(\xi)}{\mathrm{d}\xi} = 4S(\xi)k\frac{\mathrm{d}S(\xi)}{\mathrm{d}\xi}, \end{aligned} \tag{10}$$

with the condition

$$\sigma = 2\alpha. \tag{11}$$

Integrating the second equation yields

$$L(\xi) = \frac{2S^2(\xi)}{\lambda} + C, \tag{12}$$

which the first equation reduces to

$$\frac{2S^3(\xi)}{\lambda} + (C + \alpha^2 - \gamma)S(\xi) - k^2\frac{\mathrm{d}^2S(\xi)}{\mathrm{d}\xi^2} = 0, \tag{13}$$

where C is an arbitrary constant.

Note that if we obtain one solution S from (13), we can derive the corresponding solution L using (12).

Case 1: According to Step II and Step III, we assume that the solution of (13) has the form

$$\begin{aligned} S(\xi) &= F_1[\wp(\xi;g_2,g_3)] \\ &= A_0 + \frac{A_1\wp(\xi;g_2,g_3) + B_1\wp'(\xi;g_2,g_3)}{R + P\wp(\xi;g_2,g_3) + Q\wp'(\xi;g_2,g_3)}, \end{aligned} \tag{14}$$

where $\wp(\xi;g_2,g_3)$ satisfies (7) and (8), and A_0, A_1, B_1, R, P, Q are constants to be determined.

Therefore we have from (7) and (8)

$$\begin{aligned} \frac{\mathrm{d}^2S(\xi)}{\mathrm{d}\xi^2} &= \frac{1}{2(R+Pf+Q\phi)^3}\Big\{-48B_1R\wp^4Q \\ &+ \big[(4B_1P^2 - 4A_1QP)\wp' - 4A_1RP - 16B_1PQg_2 \\ &\quad + 16A_1Q^2g_2\big]\wp^3 + (-24A_1RQ + 12B_1RP)\wp' \\ &+ 36A_1Q^2g_3 + 12A_1R^2 - 36B_1Pg_3Q\wp^2 \\ &+ \big[(24B_1R^2 - B_1P^2g_2 + A_1Qg_2P)\wp' \\ &\quad - 24B_1RQg_3 + 3A_1Rg_2P\big]\wp \\ &+ (4A_1Qg_3P - 4B_1P^2g_3 + 3B_1Rg_2P)\wp' \\ &+ 4A_1RPg_3 - B_1Rg_2^2Q - A_1R^2g_2 \\ &- A_1Q^2g_3g_2 + B_1Pg_3Qg_2\Big\}. \end{aligned} \tag{15}$$

With the aid of symbolic computation (Maple), by inserting (14) into (13) along with (15) and equating the coefficients of these terms $\wp'^i\wp^j$ $(i = 0,1; j = 0,1,2,\ldots)$, we get a set of algebraic equations with respect to the unknowns $k, \lambda, A_0, A_1, B_1, R, P, Q, C$, which are complicated. Thus we omit them here. Solving the set of algebraic equations yields:

Family 1.

$$\begin{aligned} S(\xi) &= -\frac{B_1}{2Q} \\ &\quad + \frac{A_1\wp(\xi;g_2,g_3) + B_1\wp'(\xi;g_2,g_3)}{R + P\wp(\xi;g_2,g_3) + Q\wp'(\xi;g_2,g_3)}, \\ L(\xi) &= \frac{2}{\lambda}\Big[-\frac{B_1}{2Q} \\ &\quad + \frac{A_1\wp(\xi;g_2,g_3) + B_1\wp'(\xi;g_2,g_3)}{R + P\wp(\xi;g_2,g_3) + Q\wp'(\xi;g_2,g_3)}\Big]^2 + C, \\ C &= -\alpha^2 + \gamma, \\ \lambda &= \frac{B_1^3}{2(B_1P - A_1Q)Pk^2}, \\ R &= \frac{P(B_1^2P^2 - 3B_1PQA_1 + 2A_1^2Q^2)}{4B_1^2Q^2}, \\ g_2 &= \frac{P}{4B_1^3Q^4}\big(P^3B_1^3 - 4A_1P^2B_1^2Q \\ &\quad + 5A_1^2PB_1Q^2 - 2A_1^3Q^3\big), \\ g_3 &= 0, \end{aligned} \tag{16}$$

where $\alpha, \gamma, k, A_1, B_1, P, Q$ are arbitrary constants.

Family 2.

$$\begin{aligned} S(\xi) &= \frac{B_1\wp'(\xi;g_2,g_3)}{P\wp(\xi;g_2,g_3)}, \\ L(\xi) &= \frac{2}{\lambda}\left[\frac{B_1\wp'(\xi;g_2,g_3)}{P\wp(\xi;g_2,g_3)}\right]^2 + C, \\ C &= -\alpha^2 + \gamma, \\ \lambda &= \frac{4B_1^2}{k^2P^2}, \\ g_2 &= \text{arbitrary constant}, \ g_3 = 0, \end{aligned} \tag{17}$$

where $\alpha, \gamma, k, B_1, P$ are arbitrary constants.

Family 3.

$$\begin{aligned} S(\xi) &= \frac{B_1\wp'(\xi;g_2,g_3)}{R + P\wp(\xi;g_2,g_3)}, \\ L(\xi) &= \frac{2}{\lambda}\left[\frac{B_1\wp'(\xi;g_2,g_3)}{R + P\wp(\xi;g_2,g_3)}\right]^2 + C, \\ C &= -\frac{-6k^2R + \alpha^2P - \gamma P}{P}, \\ \lambda &= \frac{4B_1^2}{k^2P^2}, \quad g_2 = \frac{4R^3 + P^3g_3}{RP^2}, \\ g_3 &= \text{arbitrary constant}, \end{aligned} \tag{18}$$

where $\alpha, \gamma, k, P, R, B_1$ are arbitrary constants.

Family 4.

$$S(\xi) = A_0 + \frac{-A_0 P\wp(\xi;g_2,g_3) + B_1\wp'(\xi;g_2,g_3)}{P\wp(\xi;g_2,g_3) + Q\wp'(\xi;g_2,g_3)},$$
$$L(\xi) = \frac{2}{\lambda}\left[A_0 + \frac{-A_0 P\wp(\xi;g_2,g_3) + B_1\wp'(\xi;g_2,g_3)}{P\wp(\xi;g_2,g_3) + Q\wp'(\xi;g_2,g_3)}\right]^2 + C, \quad (19)$$
$$C = -\alpha^2 + \gamma,$$
$$\lambda = \frac{4(A_0^2Q^2 + 2B_1A_0Q + B_1^2)}{k^2P^2},$$
$$g_2 = g_3 = 0,$$

where $\alpha,\gamma,k,A_0,B_1,P,Q$ are arbitrary constants.

Family 5.

$$S(\xi) = \frac{A_1\wp(\xi;g_2,g_3)}{Q\wp'(\xi;g_2,g_3)},$$
$$L(\xi) = \frac{2}{\lambda}\left[\frac{A_1\wp(\xi;g_2,g_3)}{Q\wp'(\xi;g_2,g_3)}\right]^2 + C, \quad (20)$$
$$C = -\alpha^2 + \gamma,$$
$$g_2 = \frac{A_1^2}{4\lambda Q^2k^2},\ g_3 = 0,$$

where α,γ,k,A_1,Q are arbitrary constants.

Family 6.

$$S(\xi) = A_0 + \frac{A_1\wp(\xi;g_2,g_3)}{R + P\wp(\xi;g_2,g_3)},$$
$$L(\xi) = \frac{2}{\lambda}\left[A_0 + \frac{A_1\wp(\xi;g_2,g_3)}{R + P\wp(\xi;g_2,g_3)}\right]^2 + C,$$
$$C = -\big[5\alpha^2A_1\lambda P^2 + 6\alpha^2A_0P^3\lambda - 6\gamma A_0P^3\lambda + 30A_1A_0^2P^2 24A_1^2A_0P - 5\gamma A_1\lambda P^2 + 12A_0^3P^3 + 6A_1^3\big]\big[\lambda P^2(6A_0P + 5A_1)\big]^{-1},$$
$$R = \big[-2(A_0^3P^3 + 3A_1A_0^2P^2 + 3A_1^2A_0P + A_1^3)\big] \cdot \big[P(6A_0P + 5A_1)k^2\lambda\big]^{-1},$$
$$g_2 = 8\big[6A_0^6P^6 + 30A_1A_0^5P^5 + 61A_1^2A_0^4P^4 + 64A_1^3A_0^3P^3 + 36A_1^4A_0^2P^2 + 10A_1^5A_0P + A_1^6\big] \cdot \big[P^4\lambda^2(6A_0P + 5A_1)^2k^4\big]^{-1},$$
$$g_3 = -4\big[330A_1^6A_0^3P^3 + 16A_0^9P^9 + 120A_1A_0^8P^8 + 394A_1^2A_0^7P^7 + 741A_1^3A_0^6P^6 + 876A_1^4A_0^5P^5 + 671A_1^5A_0^4P^4 + 16A_1^8A_0P + 99A_1^7A_0^2P^2 + A_1^9\big]\big[P^6(6A_0P + 5A_1)^3k^6\lambda^3\big]^{-1}, \quad (21)$$

where $\alpha,\lambda,\gamma,k,A_0,A_1,P$ are arbitrary constants.

Case 2: According to Step II and Step III, we assume that the solution of (13) has the form

$$\begin{aligned} S(\xi) &= F_2[\wp(\xi;g_2,g_3)] \\ &= a_0 + a_1[E_1\wp(\xi;g_2,g_3) + E_2]^{1/2} \\ &\quad + b_1[E_1\wp(\xi;g_2,g_3) + E_2]^{-1/2}, \end{aligned} \quad (22)$$

where $\wp(\xi;g_2,g_3)$ satisfies (7) and (8), and a_0,a_1,E_1,E_2,b_1 are constants to be determined.

Therefore we have from (7) and (8)

$$\begin{aligned} \frac{\mathrm{d}^2S(\xi)}{\mathrm{d}\xi^2} &= \frac{E_1}{4(E_1\wp + E_2)^{5/2}}\big[-3E_1b_1g_3 + b_1g_2E_2 \\ &- 12b_1\wp^2E_2 - a_1E_2^2g_2 + 12a_1E_2^2\wp^2 + 8E_1^2a_1\wp^4 \\ &+ 20E_1a_1\wp^3E_2 - 2b_1g_2E_1\wp + E_1^2a_1g_3\wp \\ &+ E_1a_1g_3E_2 - E_1a_1\wp g_2E_2\big]. \end{aligned} \quad (23)$$

With the aid of Maple, we substitute (22) and (23) into (13) and equate the coefficients of these terms $\wp^j(\sqrt{E_1\wp + E_2})^i$ $(i = 0,1; j = 0,1,2,3,4,5)$; we get a set of nonlinear algebraic equations with respect to the unknowns E_1,E_2,a_0,a_1,b_1. By solving the set of nonlinear algebraic equations, we can determine these unknowns as follows:

Family 7.

$$S(\xi) = \frac{b_1}{\sqrt{E_1\wp(\xi;g_2,g_3) + E_2}},$$
$$L(\xi) = \frac{2}{\lambda}\left[\frac{b_1}{\sqrt{E_1\wp(\xi;g_2,g_3) + E_2}}\right]^2 + C,$$
$$C = \frac{-E_1\alpha^2 - 3k^2E_2 + E_1\gamma}{E_1}, \quad (24)$$
$$g_2 = \frac{4(-b_1^2E_1 + 3\lambda E_2^2k^2)}{E_1^2k^2\lambda},$$
$$g_3 = \frac{4E_2(2\lambda E_2^2k^2 - b_1^2E_1)}{E_1^3k^2\lambda},$$

where $\alpha,\gamma,\lambda,k,E_1,E_2,b_1$ are arbitrary constants.

Family 8.

$$S(\xi) = a_1\sqrt{\frac{k^2\lambda}{a_1^2}\wp(\xi;g_2,g_3) + E_2},$$
$$L(\xi) = \frac{2}{\lambda}\left[a_1\sqrt{\frac{k^2\lambda}{a_1^2}\wp(\xi;g_2,g_3) + E_2}\right]^2 + C,$$

$$C=\frac{\gamma\lambda-3a_1^2E_2-\alpha^2\lambda}{\lambda},$$
$$g_2=\text{arbitrary constant},$$
$$g_3=\frac{a_1^2E_2(-4a_1^4E_2^2+k^4\lambda^2g_2)}{k^6\lambda^3}, \qquad (25)$$

where $\alpha,\gamma,\lambda,k,E_2,a_1$ are arbitrary constants.

Family 9.

$$S(\xi)=a_1\sqrt{\frac{k^2\lambda}{a_1^2}\wp(\xi;g_2,g_3)+E_2}+b_1\left(\frac{k^2\lambda}{a_1^2}\wp(\xi;g_2,g_3)+E_2\right)^{-1/2},$$

$$L(\xi)=\frac{2}{\lambda}\left[a_1\sqrt{\frac{k^2\lambda}{a_1^2}\wp(\xi;g_2,g_3)+E_2}+\frac{b_1}{\sqrt{\frac{k^2\lambda}{a_1^2}\wp(\xi;g_2,g_3)+E_2}}\right]^2+C,$$
$$C=\frac{-6b_1a_1+\gamma\lambda-3a_1^2E_2-\alpha^2\lambda}{\lambda},$$
$$g_2=\frac{4a_1^2(3a_1^2E_2^2-b_1^2)}{k^4\lambda^2},$$
$$g_3=\frac{4a_1^4E_2(2a_1^2E_2^2-b_1^2)}{k^6\lambda^3}, \qquad (26)$$

where $\alpha,\gamma,\lambda,k,E_2,a_1,b_1$ are arbitrary constants.

Remark. In particular, the Weierstrass elliptic function can be written as

$$\wp(\xi;g_2,g_3)=e_2-(e_2-e_3)\mathrm{cn}^2(\sqrt{e_1-e_3}\xi;m) \qquad (27)$$

in terms of the Jacobian elliptic cosine function, where $m^2=(e_2-e_3)/(e_1-e_3)$ is the modulus of the Jacobian elliptic function; e_i $(i=1,2,3;e_1\geq e_2\geq e_3)$ are the three roots of the cubic equation $4y^3-g_2y-g_3=0$.

Therefore solutions (16) are rewritten as

$$S(\xi)=-\frac{B_1}{2Q}+\frac{A_1[e_2-N_1\mathrm{cn}^2(\sqrt{N_2}\xi;m)]+2B_1N_1\sqrt{N_2}\mathrm{cn}(\sqrt{N_2}\xi;m)\mathrm{sn}(\sqrt{N_2}\xi;m)\mathrm{dn}(\sqrt{N_2}\xi;m)}{R+P[e_2-N_1\mathrm{cn}^2(\sqrt{N_2}\xi;m)]+2QN_1\sqrt{N_2}\mathrm{cn}(\sqrt{N_2}\xi;m)\mathrm{sn}(\sqrt{N_2}\xi;m)\mathrm{dn}(\sqrt{N_2}\xi;m)},$$

$$L(\xi)=\frac{2}{\lambda}\left[-\frac{B_1}{2Q}+\frac{A_1[e_2-N_1\mathrm{cn}^2(\sqrt{N_2}\xi;m)]+2B_1N_1\sqrt{N_2}\mathrm{cn}(\sqrt{N_2}\xi;m)\mathrm{sn}(\sqrt{N_2}\xi;m)\mathrm{dn}(\sqrt{N_2}\xi;m)}{R+P[e_2-N_1\mathrm{cn}^2(\sqrt{N_2}\xi;m)]+2QN_1\sqrt{N_2}\mathrm{cn}(\sqrt{N_2}\xi;m)\mathrm{sn}(\sqrt{N_2}\xi;m)\mathrm{dn}(\sqrt{N_2}\xi;m)}\right]^2+C,$$

$$C=-\alpha^2+\gamma,$$
$$\lambda=\frac{B_1^3}{2(B_1P-A_1Q)Pk^2},$$
$$R=\frac{P}{4B_1^2Q^2}(B_1^2P^2-3B_1PQA_1+2A_1^2Q^2),$$
$$g_2=\frac{P}{4B_1^3Q^4}(P^3B_1^3-4A_1P^2B_1^2Q+5A_1^2PB_1Q^2-2A_1^3Q^3),\quad g_3=0, \qquad (28)$$

where $N_1=e_2-e_3,N_2=e_1-e_3$.

In particular, when $m\to 1$, i. e., $e_2\to e_1,\mathrm{cn}(\xi;m)\to\mathrm{sech}(\xi)$, thus the solitary wave solutions of (1) can be written as

$$S(\xi)=-\frac{B_1}{2Q}+\frac{A_1[e_2-N_1\mathrm{sech}^2(\sqrt{N_2}\xi;m)]+2B_1N_1\sqrt{N_2}\tanh(\sqrt{N_2}\xi;m)\mathrm{sech}^2(\sqrt{N_2}\xi;m)}{R+P[e_2-N_1\mathrm{sech}^2(\sqrt{N_2}\xi;m)]+2QN_1\sqrt{N_2}\mathrm{sech}^2(\sqrt{N_2}\xi;m)\tanh(\sqrt{N_2}\xi;m)},$$

$$L(\xi)=\frac{2}{\lambda}\left[-\frac{B_1}{2Q}+\frac{A_1[e_2-N_1\mathrm{sech}^2(\sqrt{N_2}\xi;m)]+2B_1N_1\sqrt{N_2}\mathrm{sech}^2(\sqrt{N_2}\xi;m)\tanh(\sqrt{N_2}\xi;m)}{R+P[e_2-N_1\mathrm{sech}^2(\sqrt{N_2}\xi;m)]+2QN_1\sqrt{N_2}\mathrm{sech}^2(\sqrt{N_2}\xi;m)\tanh(\sqrt{N_2}\xi;m)}\right]^2+C,$$

$$C = -\alpha^2 + \gamma,$$

$$\lambda = \frac{B_1^3}{2(B_1P - A_1Q)Pk^2},$$

$$R = \frac{P}{4B_1^2Q^2}(B_1^2P^2 - 3B_1PQA_1 + 2A_1^2Q^2),$$

$$g_2 = \frac{P}{4B_1^3Q^4}(P^3B_1^3 - 4A_1P^2B_1^2Q + 5A_1^2PB_1Q^2 - 2A_1^3Q^3), \quad g_3 = 0. \tag{29}$$

Similarly, we rewrite the solutions (1) as other forms in terms of the Jacobian elliptic function or the hyperbolic function.

The profile of the above solution for the parametric choice is shown in Figs. 1 and 2.

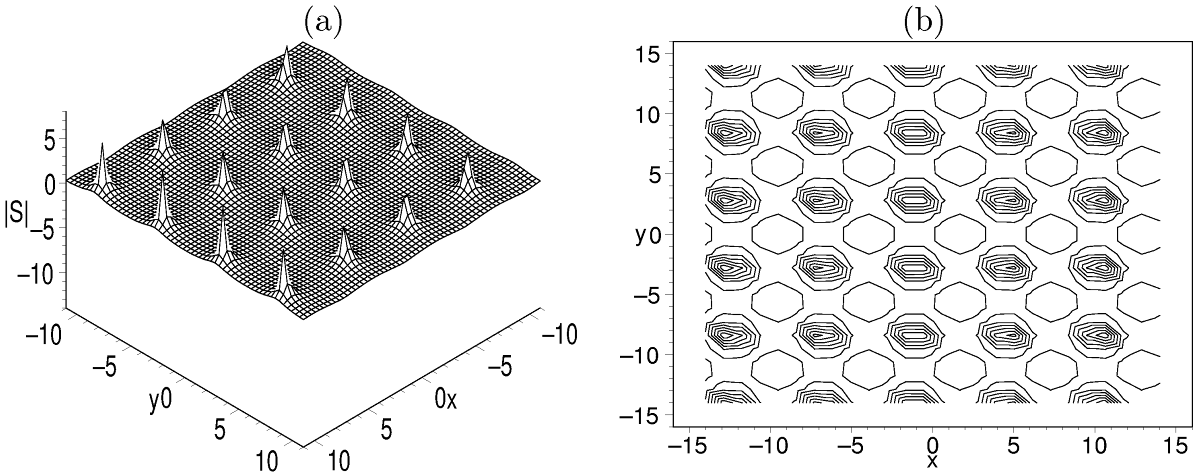

Fig. 1. (a) Intensity $|S|$ according to (16) with the parameters $Q = 2, B_1 = A_1 = P = 1$. (b) Contour of $|S|$ with the parameters of (a).

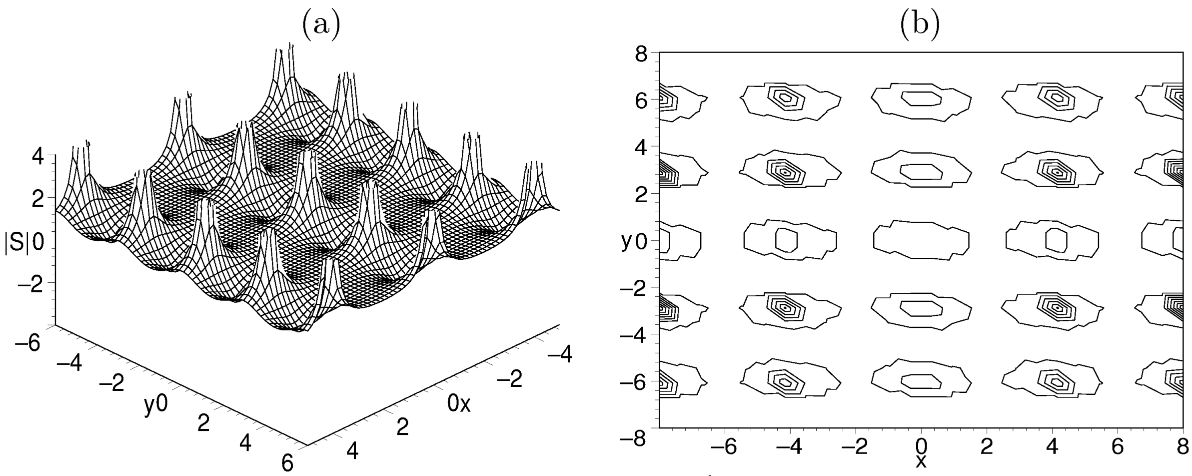

Fig. 2. (a) Intensity $|S|$ according to (26) with the parameters $b_1 = a_1 = \frac{1}{2}, k = E_2 = \alpha = \lambda = 1$. (b) Contour of $|S|$ with the parameters of (a).

4. Conclusion

In summary, we firstly transformed the LSRI system (1) into the nonlinear ordinary differential equation (10) using a series of ansatz. Then with the aid of Maple, we used a transformation in terms of the Weierstrass elliptic function to obtain nine families of doubly periodic solutions of (1), and to give figure examples of these families. In particular, new plane solitary wave solutions (a special class of solutions which do not vanish along the line $\xi = 0$ for $x, y \to \infty$) were also derived. These solutions are useful to explain the corresponding physical phenomena.

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